POWER-AWARE ROUTING IN NETWORKS

A Thesis

by

DIBAKAR DAS

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2011

Major Subject: Computer Engineering

POWER-AWARE ROUTING IN NETWORKS

A Thesis

by

DIBAKAR DAS

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Approved by:

Co-Chairs of Committee,	Narasimha Reddy
	Srinivas Shakkottai
Committee Members,	Natarajan Gautam
Head of Department,	Costas N. Georghiades

August 2011

Major Subject: Computer Engineering

ABSTRACT

Power-Aware Routing in Networks. (August 2011)

Dibakar Das, B.Tech., Indian Institute of Technology Madras, India

Co-Chairs of Advisory Committee, Dr. Srinivas Shakkottai
Dr. Narasimha Reddy

The objective of this work is to develop a scheme to minimize a combination of power consumption and congestion delay in communication networks. I model the network as a set of parallel links, with flows that are able to divide their traffic among the links available to them. Power consumption at each link is concave and increasing in the load, with a non-zero intercept at the origin corresponding to idle power consumption. I believe it is possible to minimize the overall power consumption by possibly sharing links and shutting down the idle links, as long as it does not lead to significant congestion in the network. In this project,I focus on developing incentives for flows to choose the minimum cost solution. My solutions involve two elements – (i) a mypopic and selfish controller adopted by each source, which attempts to minimize cost seen by that flow, and (ii) a pricing scheme at each link whose objective is to provide appropriate signals to the controllers at the source. I use ideas drawn from population games to choose the set of source controllers, while I experiment with using marginal costs and weighted Shapley values for the pricing scheme. I show that the weighted Shapley value as a pricing scheme is superior to that of marginal cost pricing in some simple cases.

To my Parents

TABLE OF CONTENTS

LIST OF FIGURES

FIGURE Page

FIGURE Page

CHAPTER I

INTRODUCTION

With the increasing number of networking equipments, data centers etc a significant portion of the energy is spent only to keep these components alive. As mentioned in [?], the power usage of the US network infrastructure is between 5 and 24 TWh/year. There are two aspects to the power consumption in the network. The static part is mostly independent of the work-load while the dynamic part is related to the actual processing of work which is an increasing function of the load. Assuming that we have the adequate hardware support to do so, it is therefore optimal to shut down a link in an idle state. Intuitively, this suggests that in case of flows using multiple paths, we should try to concentrate the traffic to as few links as possible. However, having more flows sharing a single link may also lead to higher congestion which makes it less profitable to use the link. We begin by considering the case of minimizing power alone, without considering congestion. We will return to the congestion aspect in Chapter VI.

The power consumption in the network is typically represented by a linearly increasing function of the load with a non-zero intercept at origin [?]. In order to ensure continuity at origin, we approximate it by a concave function that is zero at origin. In particular, we use the function

$$C(x) = x - \frac{x^2}{2} \tag{1.1}$$

The journal model is *IEEE/ACM Transactions on Networking*.

which is a concave function $\forall x \in [0,1]$ and x denotes the total traffic in the link. The optimization problem is then,

$$min \sum_l C\left(\sum_i x_i^l\right), where \quad \sum_l x_i^l \quad = \quad x_i \tag{1.2}$$

Here, we have assumed no congestion, identical cost function for all links and maximum flow of 1 units, x_i^l is the amount of flow from source i through link l.

Observation: In the solution to the above optimization problem, every flow has non-zero distribution in exactly one link.

Proof. Let, $N(i)$ denote the number of links that a flow (source) i can access and let the total number of such flows be m. Let, the distribution of flows be denoted by

$$[x_i^1, x_i^2, ...x_i^{N(i)}]^T \quad = \quad \overrightarrow{x_i} \tag{1.3}$$

and the overall feasible vector by,

$$\overrightarrow{x} \quad = \quad [\overrightarrow{x_1}^T \overrightarrow{x_2}^T .. \overrightarrow{x_m}^T]^T \tag{1.4}$$

Let us denote the corner-points of the feasible set as

$$\{v_k\}, v_i \in \{\times_{j=1}^m Y_j : Y_j \in \{x_j.I^k : N(j) \geq k \geq 1\}\} \tag{1.5}$$

where $I^k \in \mathbb{R}^{N(j)}$ is the standard unit vector that has 1 at the k-th component and zero elsewhere. So, we see that the set of feasible solutions is given as

$$\overrightarrow{x} = \sum_k \lambda_k v_k, \forall \lambda_i \in \mathbb{R} \geq 0 \forall i, \sum_k \lambda_k = 1, \tag{1.6}$$

Clearly, the function $f(\overrightarrow{x}) = -C(\overrightarrow{x})$ is convex in $\overrightarrow{x}$. So by convexity,

$$f(\sum_i \lambda_i v_i) \leq \sum_i \lambda_i f(v_i) \leq max_{i=1,2..K} f(v_i) \tag{1.7}$$

Thus, the optimal solution lies at one of the corner-points of the feasible set, which proves our claim. □

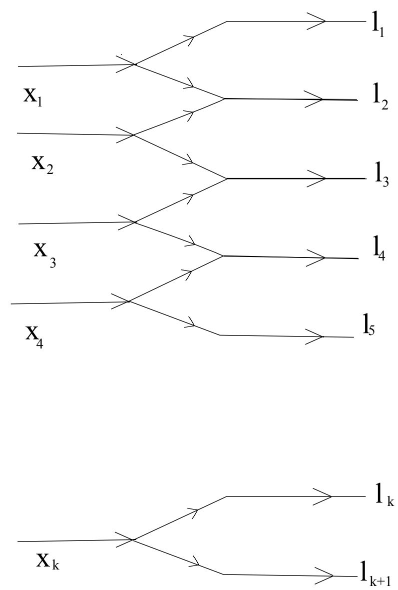

Fig. 1. A network structure with links arranged in a stack, with each flow having the choice of two links.

If the network structure is as shown in Figure 1 then the optimization problem can be converted into a minimum-weight bi-partite matching problem as shown in Figure 2. Iterative message passing has been shown to yield a correct solution to the above problem [?] [?].

For routing traffic to a link to be profitable, it requires two or more flows to simultenously share the link. In Chapter III, we use potential games and replicator dynamics and characterize the behavior of the flows in a simple 3-link scenario. We observe that, for a large number of points in the state-space, the usual pricing scheme fails to move traffic to the correct equilibrium. In Chapters IV and V, we use weighted-Shapley valued pricing schemes for the links and show that this performs better in

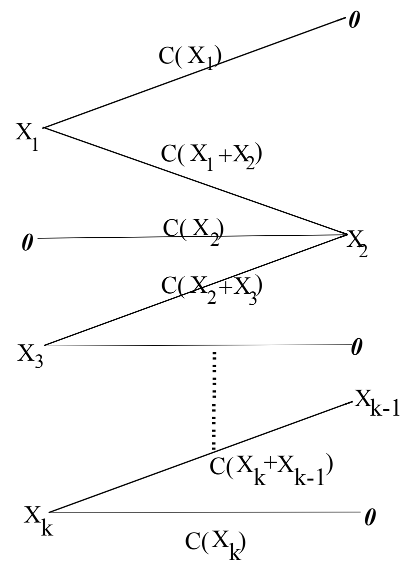

Fig. 2. The equivalent bi-partite graph, Nodes correspond to the flows in the network. A link between two nodes indicate that those two flows share a common link, **0** indicates an isolated flow. The cost vector (weights) for each combination of flows is mentioned next to each edge. The optimal solution to the k-flow problem is the minimum weight matching.

terms of the number of states that can be routed to the correct equilibrium point. In Chapter VI, we discuss the behavior of a general cost-function (both congestion and power), under replicator dynamics using marginal payoffs. Chapter VII is the conclusion.

CHAPTER II

RELATED WORK

Minimization of power consumption in the nodes through load balancing and/or sleep-time optimization is well-studied in wireless adhoc networks [?], but this is different from our work as the latter is suitable for slotted packet-transfer schemes. A related work is done in [?], where the authors have proposed a distributed joint power-delay optimization in data-centers wherein the energy-cost is a convex increasing function in the number of data-centers and non-decreasing in load, while the delay cost is a sum of queuing and network delay. The optimization is achieved by load-balancing through optimal selection of loads to be routed and the number of active servers. The idea of turning network components on or off to optimize the power consumption was studied in [?]. However, the solution provided is neither distributed, nor does it take the effect of congestion into account. Certain non-convex optimization schemes for sigmoidal curves [?] involve maximising a concavo-convex utility function while our objective is to minimize the same. In this report, we use the ideas of population games for distribution of traffic as shown in [?], along with appropriate pricing signals to achieve a low cost solution.

CHAPTER III

POPULATION GAMES

In Evolutionary game theory, we have sets of population of infinitesimal players, such that proportion of the population using a particular strategy varies over time. A population game is defined for a set $\mathbf{F} = \{1, 2, ...F\}$ of non-atomic population of players. Every infinitesimal member of the population $i \in \mathbf{F}$ can choose to take a particular action $s_i \in \mathbf{S}_i = \{1, ...S_i\}$, the set of strategies for the population i [**?**]. The population profile is then defined as the vector $\overrightarrow{x_i} = \{x_i^1, x_i^2, ...x_i^{S_i}\}$, such that $\sum_{j=1}^{S_i} x_i^j = x_i$, that indicates the probability distribution of using those strategies within population i. The set of all possible combinations of strategy distributions for a specific population is denoted by $X_i = \{\overrightarrow{x_i} \in \mathbb{R}_+^{S_i} : \sum_{j=1}^{S_i} x_i^j = x_i\}$. The state of the system is defined as the vector $\overrightarrow{X}$=$\{\overrightarrow{x_1}, \overrightarrow{x_2}, ...\overrightarrow{x_F}\}$.

The marginal payoff obtained by members of the population i by using strategy j, when the state of the system is $\mathbf{X}$ is denoted by $F_i^j(\mathbf{X}) \in \mathbb{R}$. The total payoff to the users of population i is $\sum_{j=1}^{S_i} F_i^j(\mathbf{X}) x_i^j$. When the payoff function is the marginal of a scalar cost-function that represents the energy of the system, we call it a potential game. Thus, a potential game is a population game G with a payoff function F : $\mathbf{X} \to \mathbb{R}^F$ such that there exists a continuously differentiable function $T : \mathbf{X} \to \mathbb{R}$

$$\frac{\partial T(\mathbf{X})}{\partial x_i^j} = F_i^j(\mathbf{X}), \forall i \in \mathbf{F}, j \in S_i, \tag{3.1}$$

where $X \in \mathbf{X}$ is the state of the system.

In population games, a state $\hat{x}$ is a Wardrop equilibrium if all the strategies being used results in equal marginal payoff to each member, whereas the marginal payoff that would be obtained by members is lower for all strategies not used. If $\hat{S} \in \mathbf{S}$

denotes the set of all strategies that are being used in a Wardrop equilibrium $\hat{x}$, then:

$$F_i(\hat{x}) \geq F_j(\hat{x}), \forall i \in \hat{S}, j \notin \hat{S} \tag{3.2}$$

$$F_i(\hat{x}) = F_j(\hat{x}), \forall i, j \in \hat{S} \tag{3.3}$$

Replicator dynamics are typically used to route traffic from a given state X to a Wardrop equilibrium. The dynamics, for minimizing a potential game, is given by,

$$\dot{x_i^j} = -x_i^j(F_i^j(X) - \frac{\sum_{j=1}^{S_i} F_i^j x_i^j}{x_i}) \tag{3.4}$$

At every step, the algorithm compares the fitness of a particular strategy against the average payoff of the population. The size of the members of the population using a "better" strategy continues to increase until it reaches an equilibrium such that $\dot{x_i^j} = 0, \forall i, j$. Such a point is called a fixed-point corresponding to a static population profile.

A. Modeling the problem as a routing game

We model our routing problem as a potential game G. The set of populations in the game are the set of flows $F = \{1, 2, ..N\}$. The mass of a population is then the rate of flow i and the profile is the distribution of this load among the set of all links available to it .We denote it by the vector $\overrightarrow{x_i} = \{x_i^1, x_i^2, ...x_i^{S_i}\}$. Similarly, the set of all states is denoted by $\mathbf{X}$. Corresponding to every state-flow X, the system cost function is denoted by $C(X)$ and the payoff (per unit rate) in using a link j by flow i, when the state of the system is X, is denoted by $F_i^j(X) = \frac{\partial C(\mathbf{X})}{\partial x_i^j}$

The population game is stable under replicator dynamics. The proof is similar to one given in [**?**] and hence is not repeated.

Lemma: The system of flows F that use replicator dynamics is asymptotically stable.

However, as the cost function is not convex, it is not globally asymptotically stable and there exists multiple stable local minima other than one or more global minima. The set of fixed points include both local maxima and minima, however the points corresponding to the maximas are not attained under our dynamics.

B. Example

We consider the case of two source-destination flows 1 and 2 with x_1 and x_2 as their respective amount of traffic (in $packets/sec$). The two sources use three links—1,2 and 3. Link 1 is used only by flow 1, link 3 is used solely by flow 2 while link 2 is used by both sources (as shown in Figure 3). Let, x_1^1 and x_1^2 denote the division of traffic x_1 of flow 1, such that $x_1^1 + x_1^2 = x_1$. Similarly, x_2^1 and x_2^2 denote the division of traffic x_2 of flow 2, into links 2 and 3 respectively, such that $x_2^1 + x_2^2 = x_2$. The

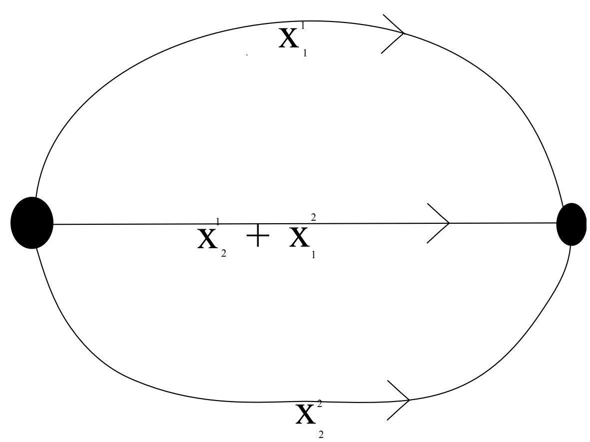

Fig. 3. Two source-destination pairs with 2 individual and one common middle link.

vector X is the state of the flows. Assuming the cost function as mentioned in the previous section, the total system cost is,

$$C(X) = (x_1^1 - \frac{(x_1^1)^2}{2}) + ((x_1^2 + x_2^1) - \frac{(x_1^2 + x_2^1)^2}{2}) + (x_2^2 - \frac{(x_2^2)^2}{2}) \tag{3.5}$$

Therefore,

$$F_1^1(X) = \frac{\partial C(X)}{\partial x_1^1} = 1 - x_1^1 \tag{3.6}$$

$$F_1^2(X) = \frac{\partial C(X)}{\partial x_1^2} = 1 - (x_1^2 + x_2^1) \tag{3.7}$$

$$F_2^1(X) = \frac{\partial C(X)}{\partial x_2^1} = 1 - (x_1^2 + x_2^1) \tag{3.8}$$

$$F_2^2(X) = \frac{\partial C(X)}{\partial x_2^2} = 1 - x_2^2. \tag{3.9}$$

Both the source nodes use their knowledge of the marginal costs to change the mass in the respective links associated with it and attempt to reach the Wardrop equilibrium, using replicator dynamics,

$$\dot{x_1^1} = x_1^1(\frac{1}{x_1}\ (x_1^1 F_1^1(X) + x_1^2 F_1^2(X)) - F_1^1(X)) \tag{3.10}$$

$$\dot{x_1^2} = x_1^2(\frac{1}{x_1}\ (x_1^1 F_1^1(X) + x_1^2 F_1^2(X)) - F_1^2(X)) \tag{3.11}$$

$$\dot{x_2^1} = x_2^1(\frac{1}{x_2}\ (x_2^1 F_2^1(X) + x_2^2 F_2^2(X)) - F_2^1(X)) \tag{3.12}$$

$$\dot{x_2^2} = x_2^2(\frac{1}{x_2}\ (x_2^1 F_2^1(X) + x_2^2 F_2^2(X)) - F_2^2(X)). \tag{3.13}$$

At equilibrium, the rate of change of traffic should be zero and thus, equating the right hand side of the equations (3.10)– (3.13) to zero, we obtain 8 possible Wardrop equilibrium states (assuming $x_1 \geq x_2$) :$(i)\{0, x_1, x_2, 0\}$, $(ii)\{x_1, 0, x_2, 0\}$, $(iii)\{x_1, 0, 0, x_2\}$, $(iv)\{\frac{x_1+x_2}{2}, \frac{x_1-x_2}{2}, x_2, 0\}$, $(v)\{\frac{x_1}{2}, \frac{x_1}{2}, 0, x_2\}$, $(vi)\{x_1, 0, \frac{x_2}{2}, \frac{x_2}{2}\}$, $(vii)\{0, x_1, 0, x_2\}$, $(viii)\{\frac{x_1+x_2}{3}, \frac{2x_1-x_2}{3}, \frac{2x_2-x_1}{3}, \frac{x_1+x_2}{3}\}$. For different initial states X, we will end up in one of the above states. Here, we have assumed that the capacity of all the links is infinite. Case (i) is the desirable state, wherein only the middle link carries the traffic of both the flows and we can turn off the two outer links. For cases (ii)–(vii), only one of the links can be turned off. Case (viii) corresponds to the global maximizer of the cost function. The equilibrium in Case (viii) is not possible if $x_2 < \frac{x_1}{2}$. Then, we can have 3 possible cases, for different values of x_1 and x_2.

Case 1: $x_1 > x_2$ and $\frac{x_1}{2} \leq x_2$

As a specific example, we choose $x_1 = .3$ and $x_2 = .2$. Then every state X can be expressed as a function of x_1^2 and x_2^1,

$$X = \{.3 - x_1^2, x_1^2, x_2^1, .2 - x_2^1\}$$

We run the system under replicator dynamics on the entire set of possible initial

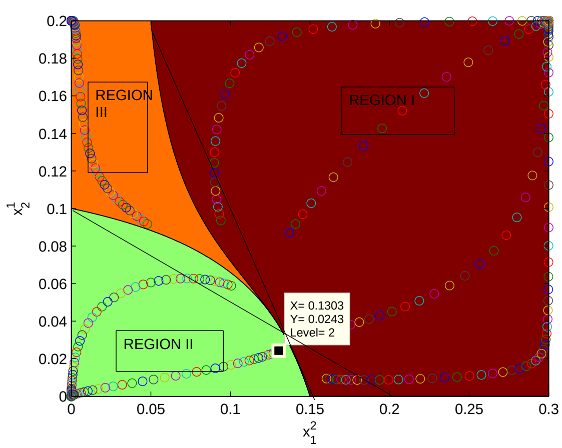

Fig. 4. Regions of convergence of traffic, when $x_1 = .3$ and $x_2 = .2$, as a function of x_1^2 and x_2^1. For initial states constituting *Region* I, the traffic is eventually carried only through the middle link. For states lying in *Region* III, the traffic is eventually carried by link 1 and 2; for states lying in *Region* II, the traffic is eventually carried by the outer links 1 and 3.

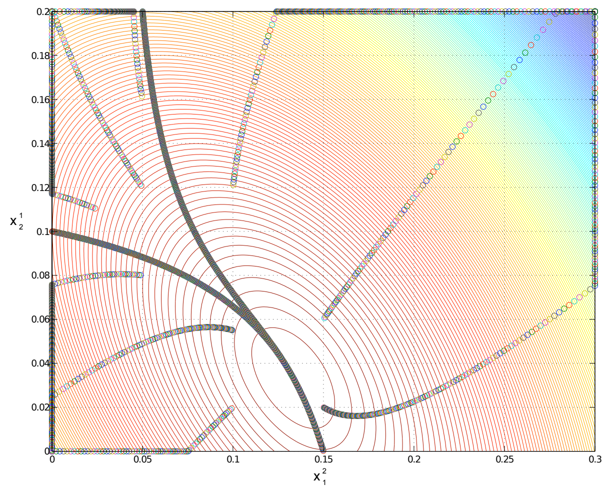

Fig. 5. Contour plots of the utility function $C(X)$,when $x_1 = .3$ and $x_2 = .2$, as a function of x_1^2 and x_2^1 . Also shown are the boundaries of the 3 regions of convergence shown in Figure 4. Also, we show the trajectories of some states using gradient descent instead of replicator dynamics.

states using step-sizes of 0.0001, as shown in Figure 4. In Figure 5, we plot the contour of the utility function $C(x)$ as a function of x_1^2 and x_2^1. We also plot the boundary of the three regions of convergence on this contour plot. In addition, on the same graph we plot the trajectory of the states using gradient descent to solve the optimization problem instead of replicator dynamics.

We made the following observations from Figure 4 and 5:

1. Out of the 8 possible Wardrop equilibria, most of the times, we reach only three of them — $\{.3, 0, 0, .2\}$, $\{0, .3, .2, 0\}$ and $\{.3, 0, 0.2, 0\}$. We can reach the other 5 equilibria points — $\{.15, .15, 0, .2\}$, $\{.3, 0, .1, .1\}$, $\{0, .3, 0, .2\}$, $\{.25, .05, .2, 0\}$ and $\{.1666, .133, .0333, .1666\}$, if and only if we start exactly at those points. Since our objective is to minimize the cost function and replicator dynamics will

continue to shift traffic as long as there is a minor difference in pay-offs, for all practical purposes, if we start with non-zero strategic distributions, we will never reach these points. The last state $\{.1666, .133, .0333, .1666\}$ is the global maximizer of the cost function wherein the load on all the links are equal.

2. *Region* I correspond to the initial states for which we reach the Wardrop equilibrium state: $\{0, .3, .2, 0\}$. This is the most desirable outcome since this Wardrop equilibrium is the global minimizer and we can switch off the two outer links. For any initial state in *Region* III, we reach the Wardrop equilibrium $\{.3, 0, .2, 0\}$ i.e. the replicator dynamics transfer all the traffic from the flow with lesser rate to the middle link while transferring all the traffic from the other flow to its outer link. For states in *Region* II, the entire flow is eventually carried only in the outer two links. We notice, that since the cost function is identical for all links, the last two Wardrop equilibriums result in the same overall cost and shutting down of one link.

3. For initial states where the load in middle link is greater than the load in both the outer two links, the pay-off in using the middle link is always lower than the pay-off in using the outer links and the dynamics is guaranteed to transfer traffic from the outer links to the middle link. That is, the sufficient conditions for the initial states to converge to the equilibrium $\{0, .3, .2, 0\}$ are given by :

$$2x_1^2 + x_2^1 \geq .3 \tag{3.14}$$

$$2x_2^1 + x_1^2 \geq .2 \tag{3.15}$$

In Figure 4, the above conditions refer to the region to the right of the two straight lines. This region is a subset of the actual set of states converging to the same equilibrium point.

4. The curves indicating the boundary between *Region* I and II, *Region* II and III, *Region* I and III all intersect at one and only one point, $\{.1667, .133, .0333, .1667\}$, which is the maximizing Wardrop equilibrium.

5. Assuming all the feasible initial states are equally likely, the probability of converging to the global optimum is, *Probability* (*Initial state in Region I*)

$$= \frac{\text{area of Region I}}{\text{total area of states}}$$

In general for any x_1 and x_2 such that, $x_1 \geq x_2$ and $\frac{x_1}{2} \leq x_2$, the lower bound to the probability of converging to the global optimum is ,

$$\begin{aligned} P &\geq \frac{5}{6} - \frac{1}{12}\left(\frac{x_2}{x_1} + \frac{x_1}{x_2}\right) \\ &\geq \frac{5}{6} - \frac{1}{12}\left(\frac{5}{2}\right) \\ &= \frac{15}{24}. \end{aligned}$$

6. As can be seen from Figure 5 the boundary between *Region* I and *Region* III is the locus of points obtained by moving along the gradient of the cost function from the singular Wardrop equilibrium $\{.25, .05, .2, 0\}$ to the maximizing equilibrium $\{.1667, .133, .0333, .1667\}$. Similarly, the boundary between *Region I* and *Region* II and *Region* II and *Region* III are obtained by moving against the gradient from the 'disjoint' Wardrop equilibrium $\{.15, .15, 0, .2\}$ to $\{.1667, .133, .0333, .1667\}$ and $\{0.3, 0, 0.1, .1\}$ to $\{.1667, .133, .0333, .1667\}$ respectively.

7. Again from Figure 5, it can be seen that the gradient descent algorithm moves the traffic to the same final equilibrium states as replicator dynamics. However, the trajectories for the convergence are not same for both.

Case 2: $x_1 = x_2$

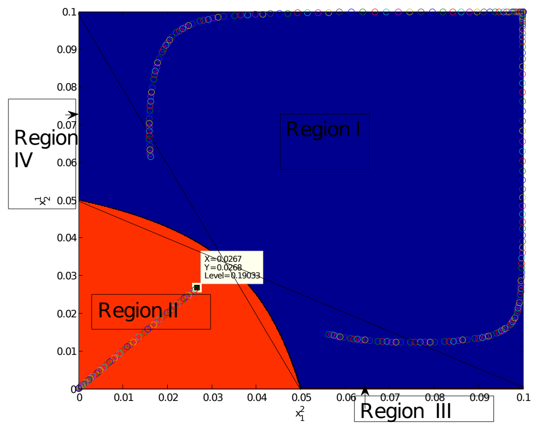

Fig. 6. Regions of convergence of traffic, when $x_1 = .1$and $x_2 = .1$, as a function of x_1^2 and x_2^1. For initial states constituting *Region* I, the traffic is eventually carried only through the middle link. For states lying in *Region* II, the traffic is eventually carried by links 1 and 3; for states lying in *Region* III, and IV the traffic is eventually carried by one outer link and one middle link.

In particular, we choose $x_1 = x_2 = .1$ (Figure 6). There are 7 possible Wardrop equilibrium states (since 2 states merge to 2 other states). Out of these- states $\{.05, .05, 0, .1\}$,

$\{0.1, 0, .05, .05\}$, $\{.0666, .0333, .0333, .0666\}$ are once again singular points and are never reached with ideal replicator dynamics.*Region* III is the region of convergence for the Wardrop equilibrium state $\{0, .1, 0, .1\}$. Similarly *Region* IV is the region of convergence for the Wardrop equilibrium $\{.1, 0, .1, 0\}$. However, we notice that both these regions consist entirely of points wherein either x_1^2 or x_2^1 is 0. So, if we start with non-zero strategy distributions, with ideal replicator dynamics, we will never end up in the Wardrop states $\{.1, 0, .1, 0\}$ and $\{0, .1, 0, .1\}$. *Region* II and *Region* I are same as before.

The lower bound on the probability of convergence to the global optimum in this case is given by,

$$P \geq 0.667$$

Case 3: $\frac{x_1}{2} \geq x_2$

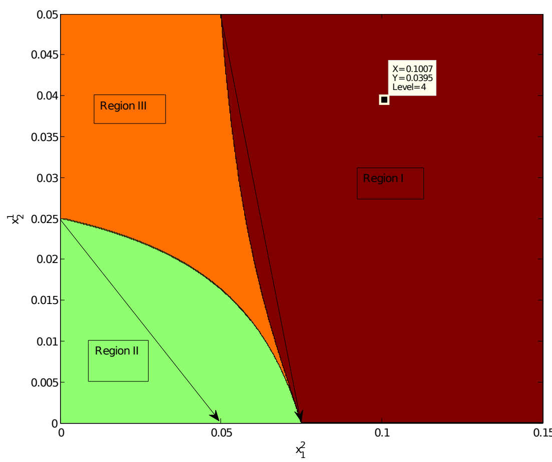

Fig. 7. Regions of convergence of traffic, when $x_1 = .15$ and $x_2 = .05$, as a function of x_1^2 and x_2^1. For initial states constituting *Region* I, the traffic is eventually carried only through the middle link. For states lying in *Region* II, the traffic is eventually carried by links 1 and 3; for states lying in *Region* III, the traffic is eventually carried by one outer link and one middle link.

We choose $x_1 = .15$ and $x_2 = .05$ for our simulations (Figure 7). Once again, there are three regions with the *Region* I corresponding to the Wardrop equilibrium with two outer links shut down, *Region* II corresponding to the two outer links being on and *Region* III corresponding to one inner link and on middle link being on. However, we notice that the sufficient condition for convergence to the middle link now is,

$$2x_1^2 + x_2^1 \geq .15 \tag{3.16}$$

Accordingly, the boundary between *Region* I and *Region* II, and the boundary between *Region* II and *Region* III touch each other at the point $(.075, 0)$. The lower bound on the probability of convergence to the global optimum in this case is given by,

$$P \geq \frac{1}{2} + \frac{1}{4}\frac{x_2}{x_1} \tag{3.17}$$

Also, if $x_2 \ll x_1$, then

$$P \approx \frac{1}{2}$$

and the lower bound approach the exact bound. Thus, we see that even in worst case, the fraction of the state-space in the three-link two-flow case that converges to the desired equilibrium state is at least $\frac{1}{2}$. In the next section, we seek to obtain an exact expression for the boundary between the different regions of convergence.

C. Approximate mathematical representation of the boundary curves

The boundary between the two regions can be obtained by using the fact that the replicator dynamics lead to the same equilibrium states as gradient descent. Therefore, using gradient descent method to minimize the cost function, the rate of change of the load in the middle link, assuming infinitesimal and equal step-size, is given by,

$$\frac{\partial x_1^2}{\partial t} = -\{1 - (x_1^2 + x_2^1)\} + \{1 - (x_1 - x_1^2)\} \quad = \quad 2x_1^2 + x_2^1 - x_1 \tag{3.18}$$

$$\frac{\partial x_2^1}{\partial t} = -\{1 - (x_1^2 + x_2^1)\} + \{1 - (x_2 - x_2^1)\} \quad = \quad 2x_2^1 + x_1^2 - x_2, \tag{3.19}$$

where x_1 and x_2 are the total throughput of the two flows. We can represent the above equation as ,

$$\dot{y} = Ay - g, \tag{3.20}$$

where $y = \begin{pmatrix} x_1^2 \\ x_2^1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $g = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We can diagonalize the matrix A as, $D = X^{-1}AX$, where D is a diagonal matrix of eigen values of A, and columns of X are the corresponding eigen-vectors. In our case, we have $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$,$X = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Multiplying both sides by X^{-1}, and denoting $X^{-1}y$ by z, $-X^{-1}g$ by g', we get $z' = Dz + g'$. Now, this differential equation pair can be solved as,

$$z_1(t) = r_1 e^{3t} - \frac{c_1'}{3} \tag{3.21}$$

$$z_2(t) = r_2 e^{t} - c_2', \tag{3.22}$$

where $z_1(t)$ and $z_2(t)$ are design parameters and $g' = \begin{pmatrix} c_1' \\ c_2' \end{pmatrix}$. Therefore,

$$x_1^2(t) = \frac{1}{\sqrt{2}}(r_1 e^{3t} - \frac{c_1'}{3} + r_2 e^{t} - c_2') \tag{3.23}$$

$$x_2^1(t) = \frac{1}{\sqrt{2}}(r_1 e^{3t} - \frac{c_1'}{3} - r_2 e^{t} + c_2') \tag{3.24}$$

When $t = 0$, we have ,

$$\begin{pmatrix} x_1^2(0) \\ x_2^1(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{c_1'}{3} + c_2' \\ \frac{c_1'}{3} - c_2' \end{pmatrix} \tag{3.25}$$

Therefore,

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \left(\begin{pmatrix} x_1^2(0) \\ x_2^1(0) \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{c_1'}{3} + c_2' \\ \frac{c_1'}{3} - c_2' \end{pmatrix} \right). \tag{3.26}$$

Equations (3.23), (3.24) and (3.26) completely define the locus of the points (x_1^2, x_2^1) under the given dynamics and initial conditions. Repeating the above process for gradient ascent, we obtain the following equations:

$$x_1^2(t) = \frac{1}{\sqrt{2}}(r_1 e^{-3t} - \frac{c_1'}{3} + r_2 e^{-t} - c_2') \tag{3.27}$$

$$x_2^1(t) = \frac{1}{\sqrt{2}}(r_1 e^{-3t} - \frac{c_1'}{3} - r_2 e^{-t} + c_2'). \tag{3.28}$$

and

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \left(\begin{pmatrix} x_1^2(0) \\ x_2^1(0) \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{c_1'}{3} + c_2' \\ \frac{c_1'}{3} - c_2' \end{pmatrix} \right) \tag{3.29}$$

The boundary between the different regions are then the trajectory of the points obtained by gradient ascent with the points corresponding to the local maxima as the initial starting point. For example- in our first example where $x_1 = .3$, $x_2 = .2$, the boundary of the three regions of convergence can be obtained as the trajectory of points under gradient ascent from initial states- $\{0.15, 0.15, 0, 2\}$, $\{0.25, 0.05, .2, 0\}$ and$\{0.3, 0, .1, .1\}$

CHAPTER IV

ROUTING USING WEIGHTED SHAPLEY VALUES FOR THE THREE LINK PROBLEM

In Game-theory, Shapley value was designed to evaluate the contribution of each individual member in a given coalition and thus attempt a fairer allocation of the net profit between the individual members.

Let's assume we have a coalition S of n members. We define a characteristic function $U(S) : 2^n \rightarrow \mathbb{R}$:

$$\begin{aligned} U(\phi) &= 0, &\quad (4.1) \\ U(A \cup B) &\geq U(A) + U(B) &\quad (4.2) \end{aligned}$$

where A and B are two disjoint coalitions. The above relations mean that the net-worth of a coalition between two disjoint coalitions is more than the sum of the two independent coalitions. Also, the net-worth of any coalition is the same independent of the way in which the coalition is formed. The allocation function $\psi : U(S) \rightarrow \mathbb{R}^n$ is then defined as,

$$\psi(U) = \{\psi_1(U), \psi_2(U), ...\psi_n(U)\} \quad (4.3)$$

Thus, the function returns the weights, called Shapley weights $\psi_i(U)$, for each member i given the overall utility function $U(S)$. Also, the sum of all $\psi_i(U)$ should be equal to the total utility $U(S)$.

The Shapley weights reflect the contribution of each participating member in the coalition. Due to property (4.2), every participating member renders a positive contribution to the coalition, the value of the coalition S with the new member minus that of the sole coalition S, i.e.

$$P(i, S) = U(S \cup \{i\}) - U(S). \tag{4.4}$$

Now, with a set F of n members, the total number of possible combinations consisting of s members is

$$\binom{n}{s}. \tag{4.5}$$

Also, the profit is shared equally among all members. So, we have our final weight factor:

$$q = \frac{1}{(s+1)\binom{n}{s}} \tag{4.6}$$

The Shapley weight for a member i is therefore,

$$\psi_i(U) = \sum_{S \subset F} qp(i, s) \tag{4.7}$$

Next, we discuss the application of Shapley value to our routing problem.

We again consider the case of a link shared by 2 flows indexed 1 and 2 with flow-rates x_1 and x_2 respectively. If $C(y)$ denotes the cost (negative utility) of using a link when the total load flowing through it is y, then the weight factor q for each flow is simply $\frac{1}{2}$ (since, there are only 2 possible ways in which flow 1 can form a coalition with flow 2 and vice versa). The overall Shapley-weight for flow 1 and 2 in this link is therefore,

$$\psi_1(C(x+y)) = \frac{1}{2}(C(x) + C(x+y) - C(y)) \tag{4.8}$$

$$\psi_2(C(x+y)) = \frac{1}{2}(C(y) + C(x+y) - C(x)). \tag{4.9}$$

Since we are interested in per-unit cost, therefore, the costs are: $\frac{1}{2x}(C(x)+C(x+y)-C(y))$ and $\frac{1}{2y}(C(y)+C(x+y)-C(x))$ for flow 1 and 2 respectively. For links that have only a single flow, the per-unit price is simply the average per-unit cost, $\frac{C(x)}{x}$ for a flow of rate x.

A. Weighted Shapley value

Instead of using constant Shapley weights of $\frac{1}{2}$, we consider the use of Shapley value of different weights such that the net-worth of the coalition still remains the same as before and is equal to the total cost of using that link. The motivation behind using a weight different from $\frac{1}{2}$ is that the weights of $\frac{1}{2}$ essentially correspond to the same dynamics as obtained using marginals. On the other hand, using different initial weights, we can control the dynamics of the state and possibly tweak it to route to a desired equilibrium point, if feasible. We use constant weights, i.e., every initial state corresponds to a single set of weights which remains unchanged throughout the rest of the dynamics.

Thus, in our 2-flow single-link, the overall Shapley weights are now:

$$\psi_{1,w}(C(x+y)) = (w_2C(x) + w_1(C(x+y) - C(y))) \tag{4.10}$$

$$\psi_{2,w}(C(x+y)) = (w_1C(y) + w_2(C(x+y) - C(x))) \tag{4.11}$$

$$w_1 + w_2 = 1, w_1, w_2 \geq 0. \tag{4.12}$$

It is easy to see that the net worth of the coalition is still the sum of the 2 individual Shapley-weights

$$\psi_{1,w}(C(x+y)) + \psi_{2,w}(C(x+y)) = C(x+y) \tag{4.13}$$

Accordingly, the per-unit prices are therefore, $\frac{1}{x}((w_2 C(x) + w_1(C(x+y) - C(y)))$ and $\frac{1}{y}((w_1 C(y) + w_2(C(x+y) - C(x)))$ respectively. The per-unit price of the flow in the single-link remains unchanged at $\frac{C(x)}{x}$. Next we discuss in detail the effect of using pricing based on Shapley-valued weights on our previous 3-link problem.

Using $C(x) = x - \frac{x^2}{2}$, the vector of prices (negative payoffs) obtained for the 3-link problem is now given by,

$$F_1^1(X) = (1 - \frac{x_1^1}{2}) \tag{4.14}$$

$$F_1^2(X) = (1 - \frac{x_1^2}{2}) - x_2^1 w_1 \tag{4.15}$$

$$F_2^1(X) = (1 - \frac{x_2^1}{2}) - x_1^2 w_2 \tag{4.16}$$

$$F_2^2(X) = (1 - \frac{x_2^2}{2}) \tag{4.17}$$

$$x_1^1 + x_1^2 = x_1, x_2^1 + x_2^2 = x_2, w_1 + w_2 = 1, w_1, w_2 \geq 0, \tag{4.18}$$

where $X = [x_1^1, x_1^2, x_2^1, x_2^2]$ denotes the current state of the system.

As a result, the sufficient conditions to route traffic corresponding to flow 1 from the link 1 to the middle link is given by,

$$F_1^1(X) - F_1^2(X) \geq 0 \tag{4.19}$$

$$i.e. \quad w_1 x_2^1 + x_1^2 \geq \frac{x_1}{2}. \tag{4.20}$$

Similarly, the sufficient condition to route traffic from link 3 to link 2 is

$$w_2 x_1^2 + x_2^1 \geq \frac{x_2}{2}. \tag{4.21}$$

Q: Find the set of sufficient conditions on x_1^2 and x_2^1 such that $\exists w_1 \in [0,1]$, which satisfies (4.20) and (4.21)

A: From (4.20), we get

$$\begin{aligned} w_1 x_2^1 &> \frac{x_1}{2} - x_1^2 \\ \Rightarrow w_1 &> \frac{(\frac{x_1}{2} - x_1^2)}{x_2^1} \end{aligned}$$

Along with $0 \leq w_1 \leq 1$, we get

$$max\{0, \frac{(\frac{x_1}{2} - x_1^2)}{x_2^1}\} \leq w_1 \tag{4.22}$$

Similarly, from (4.21), we get,

$$\begin{aligned} w_2 &\geq \frac{(\frac{x_2}{2} - x_2^1)}{x_1^2} \\ \Rightarrow (1 - w_1) &\geq \frac{(\frac{x_2}{2} - x_2^1)}{x_1^2} \\ \Rightarrow w_1 &\leq 1 - \frac{(\frac{x_2}{2} - x_2^1)}{x_1^2} \\ \Rightarrow w_1 &\leq 1 + \frac{(-\frac{x_2}{2} + x_2^1)}{x_1^2} \end{aligned}$$

So, we get the upper bound on w_1 as ,

$$w_1 \leq min\{1, 1 + \frac{(-\frac{x_2}{2} + x_2^1)}{x_1^2}\} \tag{4.23}$$

Therefore, the overall condition on w_1 is,

$$\begin{aligned} max\{0, \frac{(\frac{x_1}{2} - x_1^2)}{x_2^1}\} \leq w_1 & \\ &\leq min\{1, 1 + \frac{(-\frac{x_2}{2} + x_2^1)}{x_1^2}\} \end{aligned} \tag{4.24}$$

Now,

$$max\{0, \frac{(\frac{x_1}{2} - x_1^2)}{x_2^1}\} = \begin{cases} 0 & \text{if} \quad x_1 < 2x_1^2 \\ \frac{(\frac{x_1}{2} - x_1^2)}{x_2^1} & \text{else} \end{cases}$$

and

$$min\{1, 1+\frac{(-\frac{x_2}{2}+x_2^1)}{x_1^2}\} = \begin{cases} 1 & \text{if} \quad x_2 < 2x_2^1 \\ 1+\frac{(-\frac{x_2}{2}+x_2^1)}{x_1^2} & \text{else} \end{cases}$$

Thus, we have 4 possible cases:

Case A: $x_1^2 \geq \frac{x_1}{2}$, $x_2^1 \geq \frac{x_2}{2}$

We have $0 \leq 1$, which is satisfied for any value of x_1^2 and x_2^1 in the given range.

Case B: $x_1^2 \geq \frac{x_1}{2}$, $x_2^1 \leq \frac{x_2}{2}$

We have

$$0 \leq 1+\frac{(-\frac{x_2}{2}+x_2^1)}{x_1^2} \tag{4.25}$$

$$\Rightarrow x_1^2 + x_2^1 \geq \frac{x_2}{2} \tag{4.26}$$

Case C: $x_1^2 \leq \frac{x_1}{2}$, $x_2^1 \geq \frac{x_2}{2}$

We have

$$\frac{(\frac{x_1}{2}-x_1^2)}{x_2^1} \leq 1 \tag{4.27}$$

$$\Rightarrow \frac{x_1}{2} \leq x_2^1 + x_1^2 \tag{4.28}$$

Case D: $x_1^2 \leq \frac{x_1}{2}$, $x_2^1 \leq \frac{x_2}{2}$

We have

$$\frac{(\frac{x_1}{2}-x_1^2)}{x_2^1} \leq 1+\frac{(-\frac{x_2}{2}+x_2^1)}{x_1^2} \tag{4.29}$$

$$\Rightarrow (x_1^2-\alpha_1(x_2^1))(x_1^2-\alpha_2(x_2^1)) \geq 0, \tag{4.30}$$

where $\alpha_1(x_2^1) = \frac{-(x_2^1-\frac{x_1}{2})-\sqrt{(x_2^1-\frac{x_1}{2})^2-4({x_2^1}^2-x_2^1\frac{x_2}{2})}}{2}$ and $\alpha_2(x_2^1) = \frac{-(x_2^1-\frac{x_1}{2})+\sqrt{(x_2^1-\frac{x_1}{2})^2-4({x_2^1}^2-x_2^1\frac{x_2}{2})}}{2}$. As, $x_2^1 \leq \frac{x_2}{2}$, we have $\alpha_1(x_2^1) \geq 0$ and $\alpha_2(x_2^1) \geq 0$.

Therefore, the only possible solutions are x_1^2 such that,

$$\frac{x_1}{2} \geq x_1^2 \geq \alpha_2(x_2^1), x_2^1 \in [0, \frac{x_2}{2}] \tag{4.31}$$

We notice that for the choice of weights $w_1 = w_2 = .5$, the set of states in the state-space that converges to the optimal solution is the same as the one obtained using the marginal approach. On the other hand, varying the weights between 0 and 1 allows us to route additional states to the optimal equilibrium. Thus, we observe that the Shapley value approach performs better than the marginal approach. Also, given any initial state, we can numerically find, if possible, an appropriate weight that would make it go to the optimal state.

Since the gradient descent for a given weight vector lead to the same results as replicator dynamics, therefore,

$$\frac{\partial x_1^2}{\partial t} = (w_1 x_2^1 + x_1^2 - \frac{x_1}{2}) \tag{4.32}$$

$$\frac{\partial x_2^1}{\partial t} = (w_2 x_1^2 + x_2^1 - \frac{x_2}{2}) \tag{4.33}$$

Again, we can re-write the given systems of equations as,

$$\dot{y} = Ay - g, \tag{4.34}$$

where $y = \begin{pmatrix} x_1^2 \\ x_2^1 \end{pmatrix}$,$A = \begin{pmatrix} 1 & w_1 \\ w_2 & 1 \end{pmatrix}$, $g = \begin{pmatrix} \frac{x_1}{2} \\ \frac{x_2}{2} \end{pmatrix}$. Again, We can diagonalize the matrix A as, $D = X^{-1}AX$, where D is a diagonal matrix of eigen values of A and the columns of X are the corresponding eigen-vectors. The eigen vector matrix X is a function of w_1. Multiplying both sides by X^{-1} and denoting $X^{-1}y$ by z, $-X^{-1}g$ by g', we get

$z' = Dz + g'$ which once again can be solved as,

$$z_1(t) = r_1 e^{\lambda_1 t} - \frac{c_1'}{\lambda_1} \tag{4.35}$$

$$z_2(t) = r_2 e^{\lambda_2 t} - \frac{c_2'}{\lambda_2} \tag{4.36}$$

where $g' = \begin{pmatrix} c_1' \\ c_2' \end{pmatrix}$. Also,

$$\begin{pmatrix} x_1^2(t) \\ x_2^1(t) \end{pmatrix} = X \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \; and$$

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \frac{c_1'}{\lambda_1} \\ \frac{c_2'}{\lambda_2} \end{pmatrix} + X^{-1} \left(\begin{pmatrix} x_1^2(0) \\ x_2^1(0) \end{pmatrix} \right). \tag{4.37}$$

Also repeating the above for gradient ascent,

$$\begin{aligned} (z_2(t) + \frac{c_2'}{\lambda_2})\frac{1}{r_2} &= e^{-\lambda_2 t} \\ &= (e^{-\lambda_1 t})^{\frac{\lambda_2}{\lambda_1}} \\ &= ((z_1(t) + \frac{c_1'}{\lambda_1})\frac{1}{r_1})^{\frac{\lambda_2}{\lambda_1}} \end{aligned}$$

Lets denote $X^{-1}(w_1) = \begin{pmatrix} v_1^1 & v_1^2 \\ v_2^1 & v_2^2 \end{pmatrix}$. Then, we get the equation of the curve representing the trajectory of a given point under gradient ascent by replacing the values of z_2 and z_1 as,

$$(v_2^1 x_1^2 + v_2^2 x_2^1 + \frac{c_2'}{\lambda_2})\frac{1}{r_2} = ((v_1^1 x_1^2 + v_1^2 x_2^1 + \frac{c_1'}{\lambda_1})\frac{1}{r_1})^{\frac{\lambda_2}{\lambda_1}}$$

Again, the boundaries of the regions of convergence are the trajectories of the points obtained using gradient ascent method starting from a local maxima.

In general, for a given x_1 and x_2, and a given weight w_1, there can be 4 possible combinations of local maxima that exists. Hence there are 4 possible sets of regions of convergence maps, as described below. Again, we assume, $x_1 \geq x_2$.

Case 1:

$$\frac{x_1}{2} - w_1 x_2 \geq 0$$
$$\frac{x_2}{2} - w_2 x_1 \leq 0$$

The global maxima is once again the point of intersection of the lines: $(w_1 x_2^1 + x_1^2 -$

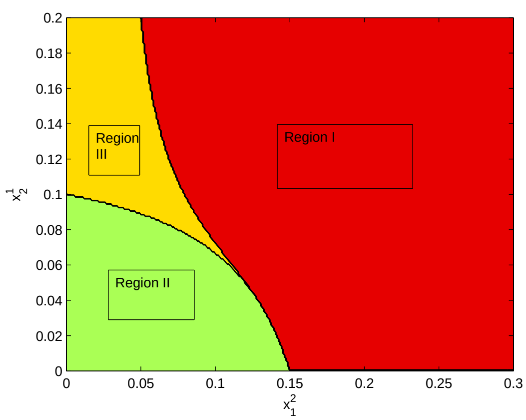

Fig. 8. Case1a:$x_1 = .3$, $x_2 = .2$, Shapley weight $w_1 = .5$. For initial states constituting *Region* I, the final equilibrium state is (.3,.2), for *Region* II :(0,0), for *Region* III, the final equilibrium state is (0,.2). Note, the graph is same as the one obtained using marginal pay-offs.

$\frac{x_1}{2}) = 0$ and $(w_2 x_1^2 + x_2^1 - \frac{x_2}{2}) = 0$. Since $x_1 \geq x_2$, it can be shown that the point of intersection lies within the feasible state-space if $(\frac{x_2}{2} - x_1 \frac{w_2}{2}) \geq 0$. Suppose that, this is true. Then the local maxima points that determine the boundary are now: $(\frac{x_1}{2} - w_1 x_2, x_2), (0, \frac{x_2}{2}), (\frac{x_1}{2}, 0)$. The feasible equilibrium points are $(0,0), (x_1, x_2)$ and

$(0, x_2)$. The boundary of the region are determined by the trajectories of the points starting from the given local maxima points (Figure 8).

If the ordinate of the point of global maxima is below zero, then we have 3 possible equilibrium states (Figure 9), but the boundary points are determined by just 2 local points within the feasible state-space: $(\frac{x_1}{2} - w_1x_2, x_2), (0, \frac{x_2}{2})$.

Case 2:

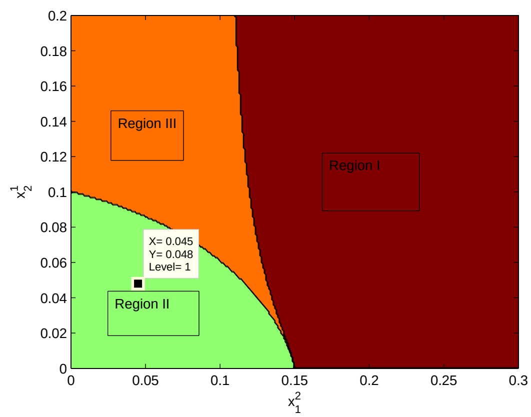

Fig. 9. Case1b:$x_1 = .3$, $x_2 = .2$, Shapley weight $w_1 = .2$. For initial states constituting *Region* I, the final equilibrium state is (.3,.2), for *Region* II :(0,0), for *Region* III, the final equilibrium state is (0,.2). The intersection of the straight-lines representing the sufficient conditions for a given w_1 is outside the state-space and thus, we have only two boundary curves.

$$\frac{x_1}{2} - w_1x_2 \geq 0$$
$$\frac{x_2}{2} - w_2x_1 \geq 0$$

Now, we have possible regions of convergence determined by the local maxima points: $(\frac{x_1}{2} - w_1x_2, x_2), (0, \frac{x_2}{2}), (\frac{x_1}{2}, 0), (x_1, \frac{x_2}{2} - w_2x_1)$. The stable equilibrium points possible are $(0, 0), (x_1, 0), (0, x_2), (x_1, x_2)$ (Figure 10).

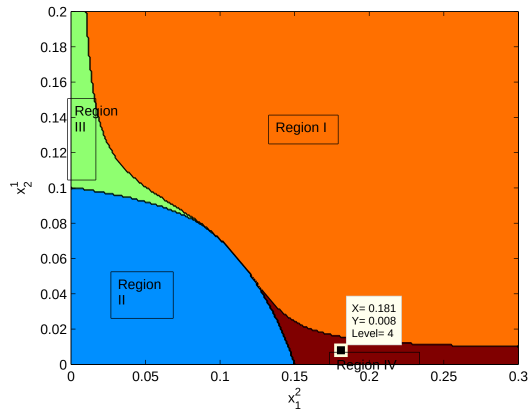

Fig. 10. Case2:$x_1 = .3$, $x_2 = .2$, Shapley weight $w_1 = .7$. For initial states constituting *Region* I, the final equilibrium state is (.3,.2), for *Region* II:(0,0), for *Region* III: (0,.2), for *Region* IV: (.3,0).

Case 3:

$$\frac{x_1}{2} - w_1 x_2 \leq 0$$
$$\frac{x_2}{2} - w_2 x_1 \geq 0$$

We have 3 regions corresponding to equilibrium points: $(0,0)$, $(x_1,0)$ and (x_1,x_2). The region corresponding to the equilibrium point $(0,x_2)$ is eliminated as for the given choice of weights, this is an unstable equilibrium (Figure 11).

Case 4:

$$\begin{aligned}
\frac{x_1}{2} - w_1 x_2 &\leq 0 \\
\frac{x_2}{2} - w_2 x_1 &\leq 0 \\
So, w_1 &\geq \frac{x_1}{2x_2} \text{and} \\
w_2 &\geq \frac{x_2}{2x_1}
\end{aligned}$$

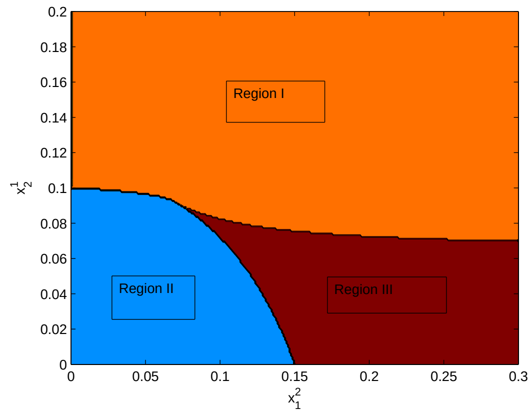

Fig. 11. Case3:$x_1 = .3$, $x_2 = .2$, Shapley weight $w_1 = .9$. For initial states constituting *Region* I, the final equilibrium state is (.3,.2), for *Region* II:(0,0), for *Region* III: (.3,0).

The only possible solution to the above set of inequalities is $x_1 = x_2$ and $w_1 = w_2 = .5$. Since, this is same as the curve obtained in the marginal case of $x_1 = x_2 = .1$, the graph is not repeated. We have 2 equilibrium stable points $(0, 0)$ and (x_1, x_2). The boundary is determined by the trajectories from the local maxima: $(0, \frac{x_1}{2})$ and $(\frac{x_1}{2}, 0)$. We make the following observations:

1. When $\frac{x_1}{2} \geq x_2$, then there is no possible weight that will eliminate the region corresponding to the undesirable equilibrium state $(0, x_2)$. Otherwise, there exist weights such that all the points in the feasible state-space goes to either $(0, 0)$ or the desirable (x_1, x_2).

2. The union of all desirable regions as weight w_1 is varied between 0 and 1 gives the set of points for which there exists a set of weights that can route the traffic to the optimal equilibrium as shown in Figure 12.

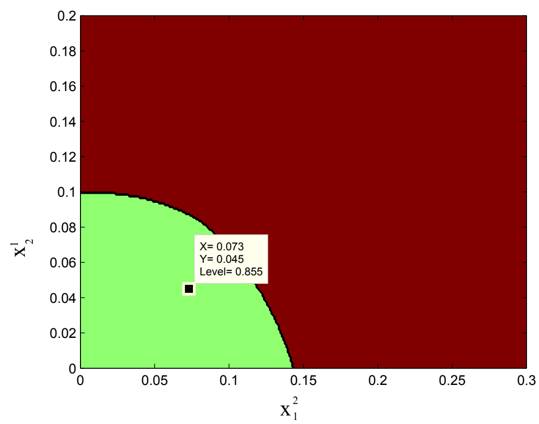

Fig. 12. With $x_1 = .3, x_2 = .2$, the regions for which there exists Shapley weights for which the initial state can be directed to (0.3,0.2) is indicated by the grey area, the area shaded green indicates points for which no such weight can be found.

3. $w_1 = w_2 = \frac{1}{2}$ corresponds to the replicator dynamics using marginal pay-offs. Thus, we claim that using the Shapley valued scheme, the region of state-space that converges to the desirable equilibrium point (x_1, x_2) is at least as large as the one obtained using marginals. Conversely, given a point (x_1^2, x_2^1), the possible weights can be found as follows. We vary the weight w_1 between 0 and 1. For a given w_1, we compute the set of feasible local maximas and hence the equation of the curves corresponding to the feasible boundaries of the regions of convergence, which are polynomials in (w_1, x_1^2, x_2^1). If the given point lies to the right and above the relevant boundary curves corresponding to one of the 4 cases as mentioned previously, we set that w_1 as our Shapley weight. Otherwise, we increase w_1. If $w_1 = 1$, then there exists no such weight for which the point can be routed to the desired equilibrium point.

CHAPTER V

SHAPLEY VALUE IN GENERAL STRUCTURES

Let us consider a simple 4-flow 5-link example. Let, the links be denoted as l_i,

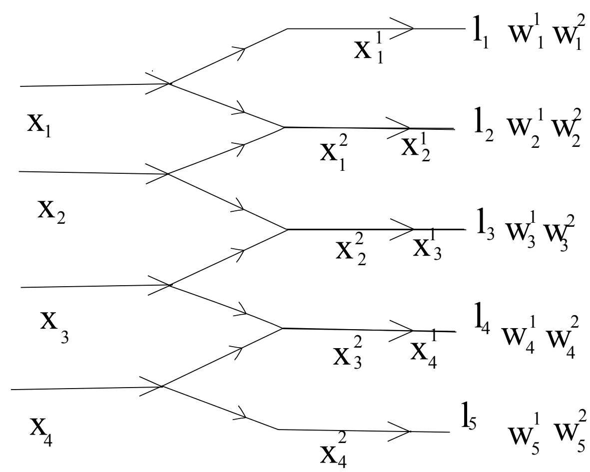

Fig. 13. 4 flows with 5 links using Shapley value.

$1 \le i \le 5$, and the flows be x_1, x_2, x_3, x_4 as shown in Figure 13. Also, from top to bottom, the contribution of flow i to the first link that it shares is denoted by x_i^1 and the one in the other link is denoted by x_i^2. Let, the weight vectors in each link be the same as shown in the figure. We notice that, $w_1^1 = 1$, $w_1^2 = 0$,$w_5^1 = 1$, $w_5^2 = 0$, since there is only one flow possible in these two "boundary links". Also, as usual,

$$w_i^1 + w_i^2 = 1, w_i^j \ge 0, \forall 1 \le i \le 5, \quad 1 \le j \le 2 \tag{5.1}$$

Now for this case, we can have only 2 possible optimum equilibriums, depending on whether (5.2) is satisfied or not:

1. Load $(x_1 + x_2)$ flowing in link l_2 and $(x_3 + x_4)$ flowing in link l_4.

2. Load $(x_2 + x_3)$ in link l_3 and x_1 in either link l_1 or l_2 and x_4 in either link l_4 or l_5. Now if the optimal equilibrium is indeed the same as in Case 1 above, then

$$\begin{aligned}
(x_1+x_2) - \frac{(x_1+x_2)^2}{2} + \\
(x_3+x_4) - \frac{(x_3+x_4)^2}{2} \\
&\le x_1 - \frac{x_1^2}{2} + (x_2+x_3) \\
&\quad -\frac{(x_2+x_3)^2}{2} + x_4 - \frac{x_4^2}{2} \\
\Rightarrow -2\frac{x_1x_2}{2} - \frac{2x_3x_4}{2} &\le \frac{-2x_2x_3}{2} \\
\Rightarrow x_1x_2 + x_3x_4 &\ge x_2x_3. \qquad (5.2)
\end{aligned}$$

On the other hand, when, $x_1x_2 + x_3x_4 \le x_2x_3$, the optimal equilibrium is the same as the one in second of the above-mentioned cases.

We consider Case 1. From (4.20) and (4.21), we see that the sufficient conditions to obtain the desired equilibrium are,

$$\begin{aligned}
(1 - \frac{x_1^1}{2}) &\ge 1 - \frac{x_1^2}{2} - w_2^1 x_2^1 \qquad (5.3)\\
(1 - \frac{x_2^1}{2} - w_2^2 x_1^2) &\le 1 - \frac{x_2^2}{2} - w_3^1 x_3^1 \qquad (5.4)\\
(1 - \frac{x_3^1}{2} - w_3^2 x_2^2) &\ge 1 - \frac{x_3^2}{2} - w_4^1 x_4^1 \qquad (5.5)\\
(1 - \frac{x_4^1}{2} - w_4^2 x_3^2) &\le 1 - \frac{x_4^2}{2}. \qquad (5.6)
\end{aligned}$$

Now, we assume that initially, all the flows have equal contribution in each of the two links that it can use, i.e. $x_i^1 = x_i^2 = \frac{x_i}{2}$. Therefore from equation 5.3, we get

$$\begin{aligned}
(1 - \frac{x_1}{4}) &\ge 1 - \frac{x_1}{4} - w_2^1 \frac{x_2}{2} \qquad (5.7)\\
\Rightarrow w_2^1 &\ge 0 \qquad (5.8)
\end{aligned}$$

Similarly, 5.4 give,

$$-w_2^2 x_1 \leq -w_3^1 x_3 \tag{5.9}$$

$$i.e. \quad w_2^2 x_1 \geq w_3^1 x_3 \tag{5.10}$$

Thus, the required sets of sufficient conditions are:

$$w_2^1 \geq 0 \tag{5.11}$$

$$w_2^2 x_1 \geq w_3^1 x_3 \tag{5.12}$$

$$w_3^2 x_2 \leq w_4^1 x_4 \tag{5.13}$$

$$w_4^2 \geq 0 \tag{5.14}$$

We choose $w_2^1, w_4^2 = \epsilon \simeq 0$.

Therefore, $w_2^2 = w_4^1 \simeq 1$.

Therefore, the required conditions are now

$$x_1 \geq w_3^1 x_3 \tag{5.15}$$

$$x_4 \geq w_3^2 x_2 \tag{5.16}$$

Now, given $x_1x_2 + x_3x_4 \geq x_2x_3$, we have, $x_1x_2 + x_3x_4 = Kx_2x_3$, where $K \geq 1$.

Therefore, $x_1x_2 = Krx_2x_3$ and $x_3x_4 = K(1-r)x_2x_3$, where $1 \geq r \geq 0$.

So, $x_1 = Krx_3$. Therefore, $x_1 \geq rx_3$ (since $K \geq 1$). Similarly, $x_4 \geq (1-r)x_2$.

Taking w_3^1=r and $w_3^2 = (1-r)$, we see that there exists weights (w_3^1, w_3^2) that satisfies equation 5.15 and 5.16. Therefore, for this 4-flow problem, given that initially all flows have equal loads in their associated links and the optimal solution is Case 1, we can find weights that will lead to the optimal solution.

Case 2: The sufficient conditions are as follows:

$$w_3^2 x_2 \quad \geq \quad w_4^1 x_4, \quad \textit{and} \tag{5.17}$$

$$w_3^1 x_3 \ \geq w_2^2 x_1 \tag{5.18}$$

Since, we do not care about the flows x_1 and x_4 (as they would concentrate to a single link, regardless of the choice of weights. This happens because the only possible stable equilibrium for these two links are one in which their entire load is in a single link), therefore, we can choose $w_4^1, w_2^2 \simeq 0$. For such a choice, any set of positive real numbers that sum to 1 would satisfy the equations 5.17 and 5.18. Thus, we see that once again there exist set of weights that concentrate the traffic flows to the optimal condition, given this specific initial condition.

Now, let us consider the following case. Assume that the desired state is as shown in

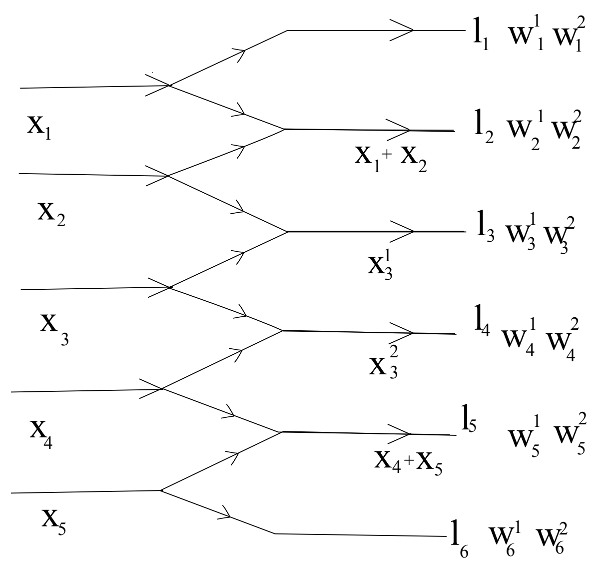

Fig. 14. Equilibrium in which a flow is not sharing a link with either of its neighboring flows.

the Figure 14. Then, we see that setting $w_4^2 = 0$ and $w_3^1 = 0$ is sufficient for flow x_4 and x_2 to re-direct traffic from l_4 to l_5 and l_3 to l_2 respectively.

In general, we can remark that if the desired equilibrium state, given this specific initial condition of "half-load", consists of such an isolated flow, then the weights corresponding to this flow can simply be chosen to be 0.

Observation: Given the initial state as mentioned before, i.e. all flows have equal loads in each of the two links that it can access, and the network structure is of k-flows with 2 links each and arranged in a stack, there exists a set of Shapley values that can re-direct traffic to the optimal state.

Proof. :Consider the following structure, the k (k is even)-flows and the corresponding weights in each of the $(k+1)$ links are shown in the Figure 15. The desired optimal state is such that every successive pair of flows is using only their common links. The sufficient conditions are given as, $w_2^1, w_k^2 \simeq 0, w_2^2 \simeq 1$, $x_1 \geq w_3^1 x_3$,
$w_4^1 x_4 \geq w_3^2 x_2$, $\quad w_4^2 x_3 \geq w_5^1 x_5, \ldots \quad w_k^1 x_k \geq w_{k-1}^2 x_{k-2}$ Now, we adopt a conservative approach to select the weights. Starting from the top, w_2^1 can be chosen close to zero, therefore $w_2^2 \simeq 1$. Selecting $w_3^1 \simeq min\{1, \frac{x_1}{x_3}\}$, ensures that $w_3^1 x_3 \leq x_1$. Now there are two possible cases:

Case 1. If $w_3^1 = 1, \quad w_3^2 = 0$. So, $w_4^1 = 0$and $w_4^2 = 1$. We select, $w_5^1 = min\{1, \frac{x_3}{x_5}\}$. Again, if $w_5^1 = 1, w_5^2 = 0$. Therefore, we can select $w_6^1 \simeq 0$, so $w_6^2 \simeq 1$ and so on.

If $w_5^1 = \frac{x_3}{x_5}$, $0 \leq w_5^2 = (1 - \frac{x_3}{x_5}) \leq 1$.

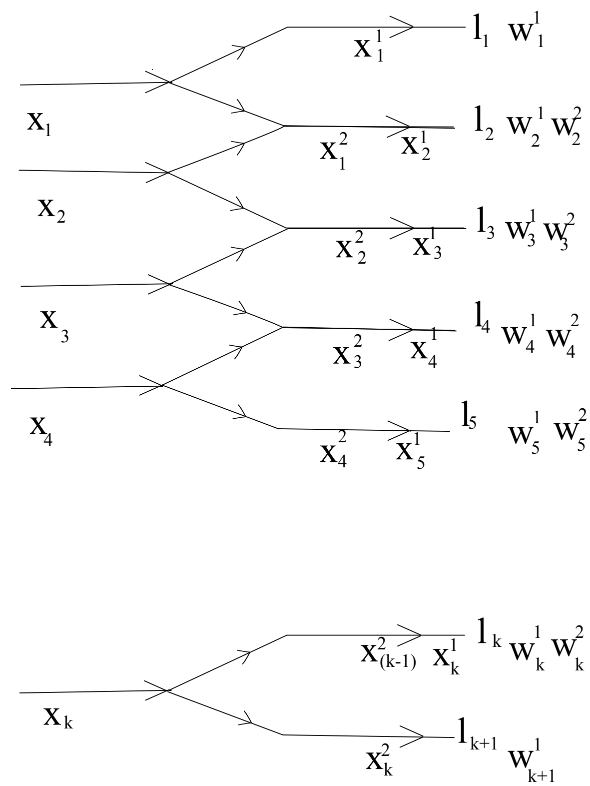

Fig. 15. k-flows with (k+1) links.

Now,

$$\begin{aligned}
w_5^2 x_4 &= (1 - \frac{x_3}{x_5})x_4 && (5.19)\\
&= \frac{(x_5 - x_3)x_4}{x_5} && (5.20)\\
&= \frac{x_5x_4 - x_3x_4}{x_5} && (5.21)\\
&\leq \frac{x_5x_6}{x_5}(since \quad x_5x_6 + x_3x_4 \geq x_5x_4) && (5.22)\\
&\leq x_6 && (5.23)
\end{aligned}$$

So, there exists $0 \leq w_6^1 \leq 1$, s.t $w_6^1 x_6 \geq w_5^2 x_4$. This process is repeated in every successive link and at n-th link, we obtain $0 \leq w_n^1, w_n^2 \leq 1$, such that the sufficient conditions till that step are satisfied.

Case 2. Similarly, if $w_3^1 = \frac{x_1}{x_3}$, then $0 \leq w_3^2 = (1 - \frac{x_1}{x_3}) \leq 1$.

Since, $x_1x_2 + x_3x_4 \geq x_2x_3$, (required for optimality), therefore, it can be shown that $w_3^2 x_2 \leq x_4$.

Therefore, there exists $0 \leq w_4^1 \leq 1$ such that, $w_4^1 x_4 \geq w_3^2 x_2$ and $0 \leq w_4^2 \leq 1$. Again, this sequence is repeated.

Thus, in a stretch of links such that every alternate link is occupied by two flows in the optimal solution, it is possible to reach those states using suitably chosen Shapley values. Also, from our previous discussion, we note that if the optimal state consists of a few isolated flows (i.e. ones which do not share their occupied link with any other flow), then we can select their corresponding Shapley weights to zero to eliminate their influence on the neighboring flows and thus separate the desirable states to continuous independent blocks of links sharing flows with neighbors separated by the isolated flows. Thus, given the optimal state and the given equal-load initial condition, it is possible to reach the optimal equilibrium using the appropriate Shapley values. □

As expected, even the "half-half" initial state does not always lead to the correct solution using marginals. eg-for a 4-flow case with $x_1 = .144, x_2 = .48, x_3 = .6, x_4 = .5$, the equilibrium reached using marginals is $[0, .144, 0, .48, 0, .6, .5, 0]$, while with weights $\{.01, .99, .2, .8, .01, .99\}$, we can reach the desired equilibrium: $[0, .144, .48, 0, 0, .6, .5, 0]$. The above observation suggests that in such cases of general links, Shapley valued pricing can perform better than marginal payoffs.

CHAPTER VI

SYSTEM WITH DELAY

We consider a delay function of form

$$\frac{Kx}{(1-x)}, \tag{6.1}$$

where K is a scalar constant. So, the overall cost function is

$$F(x) = (x - \frac{x^2}{2}) + \frac{Kx}{(1-x)}, \tag{6.2}$$

and the corresponding marginal cost function (Figure 16) is

$$f(x) = 1 - x + \frac{K}{(1-x)^2}. \tag{6.3}$$

Observation: The function F(x) is convex for $K \geq .5$

Proof. The function F(x) is convex if the second derivative $\frac{\partial^2 F}{\partial x^2} \geq 0, \forall x \in (0,1)$, i.e

$$\begin{aligned} 1 &\leq \frac{2K}{(1-x)^3} && (6.4)\\ \Rightarrow (1-x)^3 &\leq 2K && (6.5)\\ \Rightarrow 1-(2K)^{\frac{1}{3}} &\leq x && (6.6)\\ \Rightarrow K &\geq \frac{1}{2} && (6.7) \end{aligned}$$

□

The point $x_0 = 1 - (2K)^{\frac{1}{3}}$, is the minimizer of the marginal cost function $f(x)$. The constant K is chosen, such that:

a) x_0 is greater than .5,

b) The magnitude of the slope of the marginal cost curve is higher for x above x_0 than those below it.

The non-zero value of x such that $f(x) = f(0)$,$0 < x \leq 1$ is denoted by x_{cr}.

$$x_{cr}(K) \quad = \quad 1 - \frac{K}{2} - \sqrt[2]{(\frac{K^2}{4}) + K} \tag{6.8}$$

For our choice of K=0.01, the optimal solution for the previous 3-link 2-flow problem

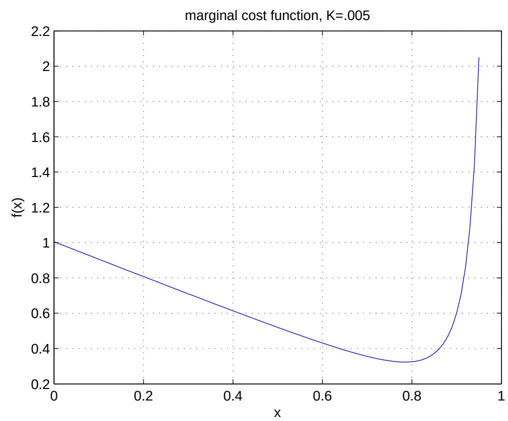

Fig. 16. Marginal cost function with K=.005.

consists of using a single link as long as the total load $x_1 + x_2 \leq x_{cr}$;to use two links when the load $x_{cr} \leq x_1 + x_2 \leq 2x_{cr}$ and to use three links for all loads greater than $2x_{cr}$. When, K=0.01, $x_0 = .7286$ and $x_{cr} = .8949$.

Claim: In the three link problem, all links are operational only when at least 2 links have loads higher than x_0, with fixed K.

Proof. We denote the load values that are greater than x_0 by x_h and the lower ones by x_l. For certain range of x_h, it is possible to find a corresponding x_l having the same marginal cost. For such a pair, let $a = -1 + \frac{2K}{(1-x_l)^3}$ and $b = -1 + \frac{2K}{(1-x_h)^3}$. Clearly, $a < 0$ and $b > 0$. Let the matrix A denote the Hessian matrix of the total system cost $C(X)$ at equilibrium. We can have three possible scenarios:

a) All links have load lower than x_l. In this case, the state vector can be written as $[x_l, x_{l1}, x_{l2}, x_l]$ where $x_{l1} + x_{l2} = x_l$. In this case, for a feasible displacement vector $d = [\epsilon, -\epsilon, 0, 0]$, we see that $d^T Ad < 0$, implying that this is not a stable equilibrium point. In fact it is a global maxima as all links with concave utility have equal load.
b) Only one link has load greater than x_0. In this case again, taking a displacement vector $d = [\epsilon, -\epsilon, \epsilon, -\epsilon]$, we find that the equilibrium point is unstable.
c)Lets take the case of two links having loads greater than x_0 and one having load x_l. Let the load vector be $[x_l, x_{h1}, x_{h2}, x_h]$, where $x_{h1} + x_{h2} = x_h$. In this case taking a general displacement vector $d = [\epsilon_1, -\epsilon_1, \epsilon_2, -\epsilon_2]$,

$$d^T Ad = a\epsilon_1^2 + b\epsilon_1\epsilon_2 + b(\epsilon_1 - \epsilon_2)^2 \tag{6.9}$$

$$\geq 0, if|b| > \frac{4|a|}{3} \tag{6.10}$$

For our choice of K=0.01, we can find a set of points x_h and x_l that satisfies the criteria: $|b| > \frac{4|a|}{3}$. Hence, there exists a stable equilibrium. □

Next, we repeat our previous simulations for the three link case using this new cost function. We have three possible cases:

A. $x_1 + x_2 \leq x_{cr}$

In particular, we take $x_1 = .3$, $x_2 = .2$. The resultant plot is shown in Figure 17. The results in this case are similar to the corresponding case when the cost function did not involve any delay element.

B. $x_{cr} \leq x_1 + x_2 \leq 2x_{cr}$

In particular, we take $x_1 = .7$ and $x_2 = .6$. The resultant plot is shown in Figure 18.

The optimal distribution is one in which there are two links active with equal

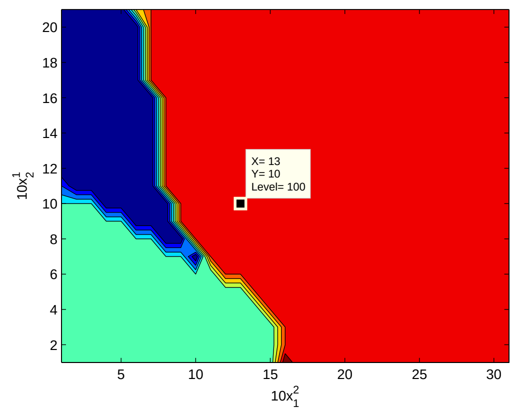

Fig. 17. Regions of convergence of traffic, when $x_1 = .3$ and $x_2 = .2$, as a function of x_1^2 and x_2^1.The regions are same as for the corresponding case without considering delays.

marginal cost, i.e with loads .8471 and .453. We see that in addition to our regular equilibrium points with zero load on the middle or "smaller" link, we have other possible distribution of loads. For low loads x_1^2 and x_2^1 on the middle link, the final equilibrium point is $[0.7, 0, 0, 0.6]$. When x_1^2 is small, but x_2^1 is high, we have another possible equilibrium $[0.7, 0, 0.6, 0]$. For high x_1^2 and low x_2^1, we have the desired equilibrium $[0, .7, .1471, .453]$ where the marginal costs corresponding to the loads in one middle link and an outer link balances each other. Similarly, we have another possible equilibrium when x_1^2 is relatively small and x_2^1 is high. The corresponding load vector $[.453, .2471, .6, 0]$ has loads in the first two links with the third link being shut off. We notice that, the equilibrium points have at most 1 link operating with load lower than x_0. The optimal solution are $[0, .7, .1471, .453]$ and $[.453, .2471, .6, 0]$ However, there are other scenarios, for higher values of $x_1 + x_2$, when we have a third link turned on, even though it is not optimal to do so. For example- when $x_1 =$

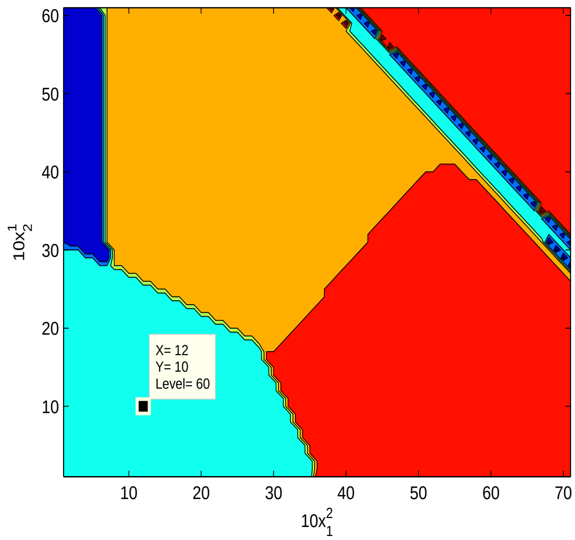

Fig. 18. Regions of convergence of traffic, when $x_1 = .7$ and $x_2 = .6$, as a function of x_1^2 and x_2^1.

$.9, x_2 = .8$, the equilibriums obtained are :$[0.85, 0.05, 0.8, 0]$,$[0.8945, 0.0055, 0, 0.8]$and $[0.0055, 0.8945, 0, 0.8]$, where the only optimal solution is: $[0.85, 0.05, 0.8, 0]$.

C. $2x_{cr} \leq x_1 + x_2 \leq 3x_{cr}$

For the third case, we take $x_1 = .95$ and $x_2 = .9$. The equilibriums obtained are: $[0.8904, 0.0096, 0.8808, 0.0692]$, $[0.8904, 0.0096, 0.0596, 0.8904]$ and$[0.0692, 0.8308, 0.0596, 0.8904]$. All of them constitute the optimal solution and all three links are active.

CHAPTER VII

CONCLUSION

In this thesis, we presented initial results of our studies on identifying the convergence region of gradient descent type controllers in problems that have non-convex objective functions. We characterized the regions of convergence to different equilibria under when considering prices based on marginal costs, and showed that performance can be poor in some cases. We also considered the case of cost plus delay in such systems, and similarly characterized the convergence regions. We also characterized the regions of convergence using Shapley value for cost functions of power and showed that through appropriate choice of Shapley weights, we can increase the region of optimal convergence. A future extension of this work could be to find an optimal scheme to choose the correct weights for the multi-link cases as well as to apply the same for a general cost function with delay.

REFERENCES

[1] S. Nedevschi, L. Popa, G. Iannaccone, S. Ratnasamy, and D. Wetherall, "Reducing network energy consumption via sleeping and rate-adaptation," in *Proc. 5th USENIX Symposium on Networked Systems Design and Implementation*, ser. NSDI'08. Berkeley, CA, USA: USENIX Association, 2008, pp. 323–336. [Online]. Available: http://portal.acm.org/citation.cfm?id=1387589.1387612.

[2] M. Bayati, D. Shah, and M. Sharma, "Maximum weight matching via max-product belief propagation," in *Proc. IEEE ISIT, 2005, pp. 1763-1767.*

[3] Y. S. Cheng, M. Neely, and K. Chugg, "Iterative message passing algorithm for bipartite maximum weighted matching," in *Proc. IEEE ISIT, 2006, pp. 1934-1938*.

[4] F. Liu, C.-Y. Tsui, and Y. J. Zhang, "Joint routing and sleep scheduling for lifetime maximization of wireless sensor networks," *IEEE Trans. Wireless Communications*, vol. 9, no. 7, pp. 2258 –2267, July 2010.

[5] Z. Liu, M. Lin, A. Wierman, S. Low, and L. Andrew, "Greening geographical load balancing," in *Proc. SIGMETRICS*, June 2011, pp. 233 − 244.

[6] L. Chiaraviglio, M. Mellia, and F. Neri, "Reducing power consumption in backbone networks," in *Proc. 2009 IEEE International Conference on Communications*, ser. ICC'09. Piscataway, NJ, USA: IEEE Press, 2009, pp. 2298–2303. [Online]. Available: http://portal.acm.org/citation.cfm?id=1817271.1817698.

[7] J.-W. Lee, R. R. Mazumdar, and N. B. Shroff, "Non-convex optimization and rate control for multi-class services in the internet," *IEEE/ACM*

Transactions Networking, vol. 13, pp. 827–840, August 2005. [Online]. Available: http://dx.doi.org/10.1109/TNET.2005.852876.

[8] V. Reddy, S. Shakkottai, A. Sprintson, and N. Gautam, "Multipath wireless network coding: A population game perspective," in *Proc. INFOCOM*, 2010, pp. 1936–1944.

VITA

Dibakar Das received the degree of Bachelor of Technology in Electrical Engineering from Indian Institute of Technology Madras,Chennai, India in July 2009. He received the Master of Science degree in Computer Engineering from Texas A&M University in August 2011. His research interests lie in communication networks, wireless ad-hoc and sensor networks.

Mr. Das may be reached at:

c/o Dr. Srinivas Shakkottai

Computer Engineering Group,

Department of Electrical & Computer Engineering,

Texas A&M University,

College Station, TX 77843-3259

Email : 952.dibakar@gmail.com